MTMA (SEMESTER-4) (COURSE: CC-9)

Lecture Notes on PDE: Wave equation

Deriving the 1D wave equation

Most of you have seen the derivation of the 1D wave equation from Newton's and Hooke's law. The key notion is that the restoring force due to tension on the string will be proportionalto the curvature at the point, as indicated in the figure. Then mass times acceleration ρu_{tt} should equal to the force, ku_{xx}. Thus $u_{tt} = c^2 u_{xx}$, where $c = \sqrt{k/\rho}$ turns out to be the velocity of the propagation.

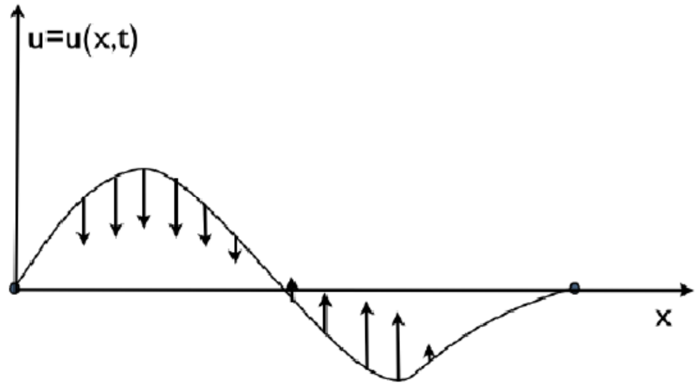


Figure 1: The restoring forces on a vibrating string, proportional to curvature.

Let's do it again, from an action integral.

Let $u = u(x,t)$ denote the deplacement of a string from the neutral position $u \equiv 0$. The mass density of the string is given by $\rho = \rho(x)$ and the elasticity given by $k = k(x)$. In paricular, in this derivation we do not assuming the the string is uniform. Consider a short piece of string, in the interval $[x, x + \Delta x]$. Its mass with be $\rho(x)\Delta x$, its velocity $u_t(x,t)$, and thus its kinetic energy, one half mass times velocity squared, is

$$\Delta K = \frac{1}{2}\rho \cdot (u_t)^2 \Delta x.$$

The total kinetic energy for the string is given by an integral,

$$K = \frac{1}{2}\int_0^L \rho \cdot (u_t)^2 \, dx.$$

From Hooke's law, the potential energy for a string is $(k/2)y^2$, where y is the length of the spring. For the stretched string, the length of the string is given by arclength $ds = \sqrt{1 + u_x^2}dx$ and so we expect a potential energy of the form

$$P = \int_0^L \frac{k}{2}(1 + u_x^2)\, dx.$$

[4] The action for a given function u is defined as the integral over time of the difference of these two energies, so

$$L(u) = \frac{1}{2}\int_0^T \int_0^L \rho \cdot (u_t)^2 - k \cdot [1 + (u_x)^2]\, dx\, dt.$$

Adding δ times a perturbation $h = h(x,t)$ to the function u gives a new action

$$L(u+\delta h) = L(u) + \delta \int_0^T \int_0^L \rho \cdot u_t \cdot h_t - k \cdot u_x h_x \, dx\, dt + \text{ higher order in } \delta.$$

The principle of least action says that in order for u to be a physical solution, the first order term should vanish for any perturbation h. Integration by parts (in t for the first term, in x for the second term, and assuming h is zero on the boundary) gives

$$0 = \int_0^T \int_0^L (-\rho \cdot u_{tt} + k \cdot u_{xx} + k_x \cdot u_x) \cdot h \, dx\, dt.$$

Since this integral is zero for all choices of h, the first factor in the integral must be zero, and we obtain the wave equation for an inhomogeneous medium,

$$\rho \cdot u_{tt} = k \cdot u_{xx} + k_x \cdot u_x.$$

When the elasticity k is constant, this reduces to usual two term wave equation

$$u_{tt} = c^2 u_{xx}$$

where the velocity $c = \sqrt{k/\rho}$ varies for changing density.

Solution of the Wave Equation by Separation of Variables:

The Problem

Let $u(x,t)$ denote the vertical displacement of a string from the x axis at position x and time t. The string has length ℓ. Its left and right hand ends are held fixed at height zero and we are told its initial configuration and speed. For notational convenience, choose a coordinate system so that the left hand end of the string is at $x = 0$ and the right hand end of the string is at $x = \ell$.

We assume that the string is undergoing small amplitude transverse vibrations so that $u(x,t)$ obeys the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \text{for all } 0 < x < \ell \text{ and } t > 0 \tag{1}$$

The conditions that the left and right hand ends are held at height zero are encoded in the "boundary conditions"

$$u(0,t) = 0 \qquad \text{for all } t > 0 \tag{2}$$
$$u(\ell,t) = 0 \qquad \text{for all } t > 0 \tag{3}$$

As we have been told the position and speed of the string at time 0, there are given functions $f(x)$ and $g(x)$ such that the "initial conditions"

$$u(x,0) = f(x) \qquad \text{for all } 0 < x < \ell \tag{4}$$
$$u_t(x,0) = g(x) \qquad \text{for all } 0 < x < \ell \tag{5}$$

are satisfied. The problem is to determine $u(x,t)$ for all x and t.

Outline of the Method of Separation of Variables

We are going to solve this problem in three steps.

Step 1 In the first step, we find all solutions of (1) that are of the special form $u(x,t) = X(x)T(t)$ for some function $X(x)$ that depends on x but not t and some function $T(t)$ that depends on t but not x. This is where the name "separation of variables" comes from. It is of course too much to expect that all solutions of (1) are of this form. But if we find a bunch of solutions $X_i(x)T_i(t)$ of this form, then since (1) is a linear equation, $\sum_i a_i X_i(x)T_i(t)$ is also a solution for any choice of the constants a_i. (Check this yourself!) If we are lucky (and we shall be lucky), we will be able to choose the constants a_i so that the other conditions (2–5) are also satisfied.

Step 2 We impose the boundary conditions (2) and (3).

Step 3 We impose the initial conditions (4) and (5).

The First Step – Finding Factorized Solutions

The factorized function $u(x,t) = X(x)T(t)$ is a solution to the wave equation (1) if and only if

$$X(x)T''(t) = c^2 X''(x)T(t) \iff \tfrac{X''(x)}{X(x)} = \tfrac{1}{c^2}\tfrac{T''(t)}{T(t)}$$

The left hand side is independent of t. So the right hand side, which is equal to the left hand side, must be independent of t too. The right hand side is independent of x. So the left hand side must be independent of x too. So both sides must be independent of both x and t. So both sides must be constant. Let's call the constant σ. So we have

$$\begin{array}{lll} & \tfrac{X''(x)}{X(x)} = \sigma & \tfrac{1}{c^2}\tfrac{T''(t)}{T(t)} = \sigma \\ \iff & X''(x) - \sigma X(x) = 0 & T''(t) - c^2\sigma T(t) = 0 \end{array} \tag{6}$$

We now have two constant coefficient ordinary differential equations, which we solve in the usual way. We try $X(x) = e^{rx}$ and $T(t) = e^{st}$ for some constants r and s to be determined. These are solutions if and only if

$$\begin{array}{lll} & \tfrac{d^2}{dx^2}e^{rx} - \sigma e^{rx} = 0 & \tfrac{d^2}{dt^2}e^{st} - c^2\sigma e^{st} = 0 \\ \iff & (r^2 - \sigma)e^{rx} = 0 & (s^2 - c^2\sigma)e^{st} = 0 \\ \iff & r^2 - \sigma = 0 & s^2 - c^2\sigma = 0 \\ \iff & r = \pm\sqrt{\sigma} & s = \pm c\sqrt{\sigma} \end{array}$$

If $\sigma \neq 0$, we now have two independent solutions, namely $e^{\sqrt{\sigma}x}$ and $e^{-\sqrt{\sigma}x}$, for $X(x)$ and two independent solutions, namely $e^{c\sqrt{\sigma}t}$ and $e^{-c\sqrt{\sigma}t}$, for $T(t)$. If $\sigma \neq 0$, the general solution to (6) is

$$X(x) = d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x} \qquad T(t) = d_3 e^{c\sqrt{\sigma}t} + d_4 e^{-c\sqrt{\sigma}t}$$

for arbitrary constants d_1, d_2, d_3 and d_4. If $\sigma = 0$, the equations (6) simplify to

$$X''(x) = 0 \qquad T''(t) = 0$$

and the general solution is

$$X(x) = d_1 + d_2 x \qquad T(t) = d_3 + d_4 t$$

for arbitrary constants d_1, d_2, d_3 and d_4. We have now found a huge number of solutions to the wave equation (1). Namely

$$\begin{array}{ll} u(x,t) = \big(d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x}\big)\big(d_3 e^{c\sqrt{\sigma}t} + d_4 e^{-c\sqrt{\sigma}t}\big) & \text{for arbitrary } \sigma \neq 0 \text{ and arbitrary } d_1, d_2, d_3, d_4 \\ u(x,t) = \big(d_1 + d_2 x\big)\big(d_3 + d_4 t\big) & \text{for arbitrary } d_1, d_2, d_3, d_4 \end{array}$$

The Second Step – Imposition of the Boundary Conditions

If $X_i(x)T_i(t)$, $i = 1, 2, 3, \cdots$ all solve the wave equation (1), then $\sum_i a_i X_i(x)T_i(t)$ is also a solution for any choice of the constants a_i. This solution satisfies the boundary condition (2) if and only if

$$\sum_i a_i X_i(0)T_i(t) = 0 \qquad \text{for all } t > 0$$

This will certainly be the case if $X_i(0) = 0$ for all i. In fact, if the a_i's are nonzero and the $T_i(t)$'s are independent, then (2) is satisfied if and only if all of the $X_i(0)$'s are zero. For us, it will be good enough to simply restrict our attention to X_i's for which $X_i(0) = 0$, so I am not even going to define what "independent" means[1]. Similarly, $u(x,t) = \sum_i a_i X_i(x) T_i(t)$ satisfies the boundary condition (3) if and only if

$$\sum_i a_i X_i(\ell) T_i(t) = 0 \qquad \text{for all } t > 0$$

and this will certainly be the case if $X_i(\ell) = 0$ for all i. We are now going to go through the solutions that we found in Step 1 and discard all of those that fail to satisfy $X(0) = X(\ell) = 0$.

First, consider $\sigma = 0$ so that $X(x) = d_1 + d_2 x$. The condition $X(0) = 0$ is satisfied if and only if $d_1 = 0$. The condition $X(\ell) = 0$ is satisfied if and only if $d_1 + \ell d_2 = 0$. So the conditions $X(0) = X(\ell) = 0$ are both satisfied only if $d_1 = d_2 = 0$, in which case $X(x)$ is identically zero. There is nothing to be gained by keeping an identically zero $X(x)$, so we discard $\sigma = 0$ completely.

Next, consider $\sigma \neq 0$ so that $d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x}$. The condition $X(0) = 0$ is satisfied if and only if $d_1 + d_2 = 0$. So we require that $d_2 = -d_1$. The condition $X(\ell) = 0$ is satisfied if and only if

$$0 = d_1 e^{\sqrt{\sigma}\ell} + d_2 e^{-\sqrt{\sigma}\ell} = d_1\left(e^{\sqrt{\sigma}\ell} - e^{-\sqrt{\sigma}\ell}\right)$$

If d_1 were zero, then $X(x)$ would again be identically zero and hence useless. So instead, we discard any σ that does not obey

$$e^{\sqrt{\sigma}\ell} - e^{-\sqrt{\sigma}\ell} = 0 \iff e^{\sqrt{\sigma}\ell} = e^{-\sqrt{\sigma}\ell} \iff e^{2\sqrt{\sigma}\ell} = 1$$

In the last step, we multiplied both sides of $e^{\sqrt{\sigma}\ell} = e^{-\sqrt{\sigma}\ell}$ by $e^{\sqrt{\sigma}\ell}$. One σ that obeys $e^{2\sqrt{\sigma}\ell} = 1$ is $\sigma = 0$. But we are now considering only $\sigma \neq 0$. Fortunately, there are infinitely many complex numbers[2] that work. In fact $e^{2\sqrt{\sigma}\ell} = 1$ if and only if there is an integer k such that

$$2\sqrt{\sigma}\ell = 2k\pi\imath \iff \sqrt{\sigma} = k\tfrac{\pi}{\ell}\imath \iff \sigma = -k^2\tfrac{\pi^2}{\ell^2}$$

With $\sqrt{\sigma} = k\frac{\pi}{\ell}\imath$ and $d_2 = -d_1$,

$$\begin{aligned}
X(x)T(t) &= d_1\left(e^{\imath\frac{k\pi}{\ell}x} - e^{-\imath\frac{k\pi}{\ell}x}\right)\left(d_3 e^{\imath\frac{ck\pi}{\ell}t} + d_4 e^{-\imath\frac{ck\pi}{\ell}t}\right) \\
&= 2\imath d_1 \sin\left(\tfrac{k\pi}{\ell}x\right)\left[(d_3 + d_4)\cos\left(\tfrac{ck\pi}{\ell}t\right) + \imath(d_3 - d_4)\sin\left(\tfrac{ck\pi}{\ell}t\right)\right] \\
&= \sin\left(\tfrac{k\pi}{\ell}x\right)\left[\alpha_k \cos\left(\tfrac{ck\pi}{\ell}t\right) + \beta_k \sin\left(\tfrac{ck\pi}{\ell}t\right)\right]
\end{aligned}$$

where $\alpha_k = 2\imath d_1(d_3 + d_4)$ and $\beta_k = -2d_1(d_3 - d_4)$. Note that, to this point, d_1, d_3 and d_4 are allowed to be any complex numbers so that α_k and β_k are allowed to be any complex numbers.

The Third Step – Imposition of the Initial Conditions

We now know that

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\tfrac{k\pi}{\ell}x\right)\left[\alpha_k \cos\left(\tfrac{ck\pi}{\ell}t\right) + \beta_k \sin\left(\tfrac{ck\pi}{\ell}t\right)\right]$$

obeys the wave equation (1) and the boundary conditions (2) and (3), for any choice of the constants α_k, β_k. It remains only to see if we can choose the α_k's and β_k's to satisfy

$$f(x) = u(x,0) = \sum_{k=1}^{\infty} \alpha_k \sin\left(\tfrac{k\pi}{\ell}x\right) \tag{4'}$$

$$g(x) = u_t(x,0) = \sum_{k=1}^{\infty} \beta_k \tfrac{ck\pi}{\ell} \sin\left(\tfrac{k\pi}{\ell}x\right) \tag{5'}$$

But any (reasonably smooth) function, $h(x)$, defined on the interval $0 < x < \ell$, has a unique representation[3]

$$h(x) = \sum_{k=1}^{\infty} b_k \sin\tfrac{k\pi x}{\ell} \tag{7}$$

as a linear combination of $\sin\frac{k\pi x}{\ell}$'s and we also know the formula

$$b_k = \tfrac{2}{\ell}\int_0^\ell h(x)\sin\tfrac{k\pi x}{\ell}\ dx$$

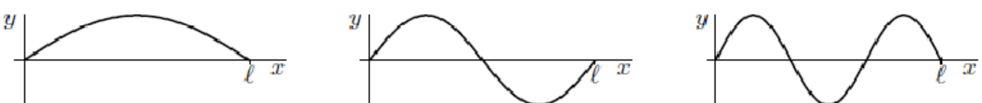
for the coefficients. We can make (7) match (4′) by choosing $h(x) = f(x)$ and $b_k = \alpha_k$. This tells us that $\alpha_k = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{k\pi x}{\ell}\ dx$. Similarly, we can make (7) match (5′) by choosing $h(x) = g(x)$ and $b_k = \beta_k\frac{ck\pi}{\ell}$. This tells us that $\frac{ck\pi}{\ell}\beta_k = \frac{2}{\ell}\int_0^\ell g(x)\sin\frac{k\pi x}{\ell}\ dx$. So we have a solution:

$$\boxed{u(x,t) = \sum_{k=1}^\infty \sin\left(\tfrac{k\pi}{\ell}x\right)\left[\alpha_k\cos\left(\tfrac{ck\pi}{\ell}t\right) + \beta_k\sin\left(\tfrac{ck\pi}{\ell}t\right)\right]} \tag{8}$$

with

$$\boxed{\alpha_k = \tfrac{2}{\ell}\int_0^\ell f(x)\sin\tfrac{k\pi x}{\ell}\ dx \qquad \beta_k = \tfrac{2}{ck\pi}\int_0^\ell g(x)\sin\tfrac{k\pi x}{\ell}\ dx}$$

While the sum (8) can be very complicated, each term, called a "mode", is quite simple. For each fixed t, the mode $\sin\left(\frac{k\pi}{\ell}x\right)\left[\alpha_k\cos\left(\frac{kc\pi}{\ell}t\right) + \beta_k\sin\left(\frac{kc\pi}{\ell}t\right)\right]$ is just a constant times $\sin\left(\frac{k\pi}{\ell}x\right)$. As x runs from 0 to ℓ, the argument of $\sin\left(\frac{k\pi}{\ell}x\right)$ runs from 0 to $k\pi$, which is k half–periods of sin. Here are graphs, at fixed t, of the first three modes, called the fundamental tone, the first harmonic and the second harmonic.



fundamental 1st harmonic 2nd harmonic

For each fixed x, the mode $\sin\left(\frac{k\pi}{\ell}x\right)\left[\alpha_k\cos\left(\frac{kc\pi}{\ell}t\right) + \beta_k\sin\left(\frac{kc\pi}{\ell}t\right)\right]$ is just a constant times $\cos\left(\frac{kc\pi}{\ell}t\right)$ plus a constant times $\sin\left(\frac{kc\pi}{\ell}t\right)$. As t increases by one second, the argument, $\frac{kc\pi}{\ell}t$, of both $\cos\left(\frac{kc\pi}{\ell}t\right)$ and $\sin\left(\frac{kc\pi}{\ell}t\right)$ increases by $\frac{kc\pi}{\ell}$, which is $\frac{kc}{2\ell}$ cycles (i.e. periods). So the fundamental oscillates at $\frac{c}{2\ell}$cps, the first harmonic oscillates at $2\frac{c}{2\ell}$cps, the second harmonic oscillates at $3\frac{c}{2\ell}$cps and so on. If the string has density ρ and tension T, then we have seen[4] that $c = \sqrt{\frac{T}{\rho}}$. So to increase the frequency of oscillation of a string you increase the tension and/or decrease the density and/or shorten the string.

Example 1 As a concrete example, suppose that

$$\begin{aligned}
&\tfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\,\tfrac{\partial^2 u}{\partial x^2}(x,t) && \text{for all } 0 < x < 1 \text{ and } t > 0\\
&u(0,t) = u(1,t) = 0 && \text{for all } t > 0\\
&u(x,0) = x(1-x) && \text{for all } 0 < x < 1\\
&u_t(x,0) = 0 && \text{for all } 0 < x < 1
\end{aligned}$$

This is a special case of equations (1–5) with $\ell = 1$, $f(x) = x(1-x)$ and $g(x) = 0$. So, by (8),

$$u(x,y) = \sum_{k=1}^\infty \sin(k\pi x)\left[\alpha_k\cos(ck\pi t) + \beta_k\sin(ck\pi t)\right]$$

with

$$\alpha_k = 2\int_0^1 x(1-x)\sin(k\pi x)\ dx \qquad \beta_k = 2\int_0^1 0\ \sin(k\pi x)\ dx = 0$$

Using[5]

$$\begin{aligned}
\int_0^1 x\sin(k\pi x)\ dx &= \int_0^1 -\tfrac{1}{\pi}\tfrac{d}{dk}\cos(k\pi x)\ dx \quad = -\tfrac{1}{\pi}\tfrac{d}{dk}\int_0^1 \cos(k\pi x)\ dx \quad = -\tfrac{1}{\pi}\tfrac{d}{dk}\tfrac{1}{k\pi}\sin(k\pi x)\Big|_0^1\\
&= -\cos(k\pi)\tfrac{1}{k\pi}\\
\int_0^1 x^2\sin(k\pi x)\ dx &= \int_0^1 -\tfrac{1}{\pi^2}\tfrac{d^2}{dk^2}\sin(k\pi x)\ dx = -\tfrac{1}{\pi^2}\tfrac{d^2}{dk^2}\int_0^1 \sin(k\pi x)\ dx = \tfrac{1}{\pi^2}\tfrac{d^2}{dk^2}\tfrac{1}{k\pi}\cos(k\pi x)\Big|_0^1\\
&= \cos(k\pi)\tfrac{2-k^2\pi^2}{k^3\pi^3} - \tfrac{2}{k^3\pi^3}
\end{aligned}$$

we have

$$\alpha_k = 2\int_0^1 x(1-x)\sin(k\pi x)\,dx = 2\big[-\cos(k\pi)\tfrac{1}{k\pi} - \cos(k\pi)\tfrac{2-k^2\pi^2}{k^3\pi^3} + \tfrac{2}{k^3\pi^3}\big] = \tfrac{4}{k^3\pi^3}[1-\cos(k\pi)]$$
$$= \begin{cases} \frac{8}{k^3\pi^3} & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

and

$$u(x,y) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \tfrac{8}{k^3\pi^3}\sin(k\pi x)\cos(ck\pi t)$$

Example 2 As a second concrete example, suppose that

$$\begin{aligned}
&\tfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\,\tfrac{\partial^2 u}{\partial x^2}(x,t) && \text{for all } 0 < x < 1 \text{ and } t > 0 \\
&u(0,t) = u(1,t) = 0 && \text{for all } t > 0 \\
&u(x,0) = \sin(5\pi x) + 2\sin(7\pi x) && \text{for all } 0 < x < 1 \\
&u_t(x,0) = 0 && \text{for all } 0 < x < 1
\end{aligned}$$

This is again a special case of equations (1–5) with $\ell = 1$. So, by (8),

$$u(x,y) = \sum_{k=1}^{\infty} \sin(k\pi x)\big[\alpha_k\cos(ck\pi t) + \beta_k\sin(ck\pi t)\big]$$

This time it is very inefficient to use the integral formulae to evaluate α_k and β_k. It is easier to observe directly, just by matching coefficients, that

$$\sin(5\pi x) + 2\sin(7\pi x) = u(x,0) = \sum_{k=1}^{\infty}\alpha_k\sin(k\pi x) \quad \Rightarrow \alpha_k = \begin{cases} 1 & \text{if } k=5 \\ 2 & \text{if } k=7 \\ 0 & \text{if } k \neq 5,7 \end{cases}$$

$$0 = u_t(x,0) = \sum_{k=1}^{\infty} ck\pi\beta_k\sin(k\pi x) \Rightarrow \beta_k = 0$$

So

$$u(x,y) = \sin(5\pi x)\cos(5c\pi t) + 2\sin(7\pi x)\cos(7c\pi t)$$

Using Fourier Series to Solve the Wave Equation

We can also use Fourier series to derive the solution (8) to the wave equation (1) with boundary conditions (2,3) and initial conditions (4,5). The basic observation is that, for each fixed $t \geq 0$, the unknown $u(x,t)$ is a function of the one variable x and this function vanishes at $x = 0$ and $x = \ell$. Thus, by the Fourier series theorem and the odd periodic extension trick, $u(x,t)$ has, for each fixed t, a unique expansion

$$u(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin\left(\tfrac{k\pi x}{\ell}\right) \tag{9}$$

By using other periodic extensions, like the even periodic extension, we can get other expansions of $u(x,t)$ too. But the odd periodic expansion (9) is particularly useful because, with it, the boundary conditions (2,3) are automatically satisfied. If we substitute $x = 0$ into the right hand side of (9) we necessarily get zero, regardless of the value of $b_k(t)$, because every term contains a factor of $\sin(0) = 0$. Similarly, if we substitute $x = \ell$ into the right hand side of (9) we again necessarily get zero, for any $b_k(t)$, because $\sin(k\pi) = 0$ for every integer k.

The solution $u(x,t)$ is completely determined by the, as yet unknown, coefficients $b_k(t)$. Furthermore these coefficients can be found by substituting $u(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin\left(\frac{k\pi x}{\ell}\right)$ into the three remaining requirements (1), (4), (5) on $u(x,t)$. First the wave equation (1):

$$0 = \frac{\partial^2 u}{\partial t^2}(x,t) - c^2\frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{k=1}^{\infty} b_k''(t)\sin\left(\frac{k\pi x}{\ell}\right) + \sum_{k=1}^{\infty}\frac{k^2\pi^2c^2}{\ell^2}b_k(t)\sin\left(\frac{k\pi x}{\ell}\right)$$
$$= \sum_{k=1}^{\infty}\left[b_k''(t) + \frac{k^2\pi^2c^2}{\ell^2}b_k(t)\right]\sin\left(\frac{k\pi x}{\ell}\right)$$

This says that, for each fixed $t \geq 0$, the function 0, viewed as a function of x, has Fourier series expansion $\sum_{k=1}^{\infty}\left[b_k''(t) + \frac{k^2\pi^2c^2}{\ell^2}b_k(t)\right]\sin\left(\frac{k\pi x}{\ell}\right)$. Applying (9) with $h(x)$ being the zero function and with b_k replaced by $\left[b_k''(t) + \frac{k^2\pi^2c^2}{\ell^2}b_k(t)\right]$ then forces

$$b_k''(t) + \frac{k^2\pi^2c^2}{\ell^2}b_k(t) = \frac{2}{\ell}\int_0^{\ell} 0\sin\left(\frac{k\pi x}{\ell}\right)dx = 0 \qquad \text{for all } k,t \tag{1'}$$

Substituting into (4) and (5) gives

$$u(0,t) = \sum_{k=1}^{\infty} b_k(0)\sin\left(\frac{k\pi x}{\ell}\right) = f(x)$$
$$\frac{\partial u}{\partial t}(0,t) = \sum_{k=1}^{\infty} b_k'(0)\sin\left(\frac{k\pi x}{\ell}\right) = g(x)$$

By uniqueness of Fourier coefficients, once again,

$$b_k(0) = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{k\pi x}{\ell}\right)dx \tag{4'}$$
$$b_k'(0) = \frac{2}{\ell}\int_0^{\ell} g(x)\sin\left(\frac{k\pi x}{\ell}\right)dx \tag{5'}$$

For each fixed k, equations (1′), (4′) and (5′) constitute one second order constant coefficient ordinary differential equation and two initial conditions for the unknown function $b_k(t)$.

You already know how to solve constant coefficient ordinary differential equations. The function $b_k(t) = e^{rt}$ satisfies the ordinary differential equation (1′) if and only if

$$r^2 + \frac{k^2\pi^2c^2}{\ell^2} = 0$$

which in turn is true if and only if

$$r = \pm i\frac{k\pi c}{\ell}$$

so that the general solution to (1′) is

$$b_k(t) = C_k e^{i\frac{k\pi c}{\ell}t} + D_k e^{-i\frac{k\pi c}{\ell}t}$$

with C_k and D_k arbitrary constants. Using $e^{\pm i\frac{k\pi c}{\ell}t} = \cos\left(\frac{k\pi c}{\ell}t\right) \pm i\sin\left(\frac{k\pi c}{\ell}t\right)$ we may rewrite the solution as

$$b_k(t) = \alpha_k\cos\left(\frac{ck\pi}{\ell}t\right) + \beta_k\sin\left(\frac{ck\pi}{\ell}t\right)$$

with $\alpha_k = C_k + D_k$ and $\beta_k = iC_k - iD_k$ again arbitrary constants. They are determined by the initial conditions (4′) and (5′).

$$(4') \implies b_k(0) = \alpha_k \quad = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{k\pi x}{\ell}\right)dx$$
$$(5') \implies b_k'(0) = \frac{ck\pi}{\ell}\beta_k = \frac{2}{\ell}\int_0^{\ell} g(x)\sin\left(\frac{k\pi x}{\ell}\right)dx \implies \beta_k = \frac{2}{ck\pi}\int_0^{\ell} g(x)\sin\left(\frac{k\pi x}{\ell}\right)dx$$

This gives us the solution (8) once again.

<u>Classification of second order, linear PDEs:</u>

A second order linear PDE in two variables x, t is an equation of the form

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_y + Fu = G,$$

where the coefficients A, B, C, D, E, F, G are constants, or specified functions of the variables x, t. The equation is classified into one of three types, based on the coefficients A, B, C, as

- *Elliptic:* if $B^2 - 4AC < 0$;
- *Parabolic:* if $B^2 - 4AC = 0$;
- *Hyperbolic:* if $B^2 - 4AC > 0$.

So for instance, Laplace's equation is elliptic, the heat equation is parabolic, and the wave equation is hyperbolic. It is useful to classify equations because the solution techniques, and properties of the solutions are different, depending on whether the equation is elliptic, parabolic, or hyperbolic. Also, the physical nature of the corresponding problems are different. For instance, elliptic equations often arise in steady-state and equilibrium problems; parabolic equations arise in diffusion problems; hyperbolic problems arise in wave motion and vibrational problems.

An equation can be of mixed type if it changes from one type to another, depending on the value of the functions A,B,C. For instance, the equation

$$tu_{xx} + u_{tt} = 0$$

is of mixed type, for $B^2 - 4AC = -4t$ is zero along the line $t = 0$ (parabolic), is positive for $t < 0$ (hyperbolic), and negative for $t > 0$ (elliptic).

When A,B,C are constant, it is always possible to make a linear change of variables to put the equation in a canonical form. This is result is as simple as diagonalizing a 2 by 2 symmetric matrix. The canonical forms as

- *Elliptic:* $u_{xx} + u_{tt} = G(x, y, u, u_t, u_x)$;
- *Parabolic:* $u_{xx} = G(x, y, u, u_t, u_x)$;;
- *Hyperbolic:* $u_{xx} - u_{tt} = G(x, y, u, u_t, u_x)$ or $u_{xt} = G(x, y, u, u_t, u_x)$.

The form $B^2 - 4AC$ is reminiscent of the quadratic formula, but it really should make you think of the determinant of the matrix

$$\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}$$

where the sign of the determinant tells you whether there are two non-zero eigenvalues of the same sign (elliptic), opposite sign (hyperbolic), or one zero eigenvalue (parabolic). This is the key to understanding the classification for linear PDEs with more variables.

For a function $u = u(x_1, x_2, x_3, \ldots, x_n)$ of n independent variables, the general linear second order PDE will be of the form

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=i}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = D$$

where the coefficients A_{ij}, B_i, C, D are constants or functions only of the independent variables. The matrix

$$\mathbf{A} = [A_{ij}]$$

can be chosen symmetric. The equation is then classified into four types, as

- *Elliptic*: if all the eigenvalues of A are nonzero, and of the same sign;
- *Parabolic*: if exactly one of the eigenvalues is zero, and the rest have the same sign;
- *Hyperbolic*: if n − 1 of the eigenvalues are of the same sign, the other of opposite sign;
- *Ultrahyperbolic*: If at least two eigenvalues are positive, at least two negative, and none are zero.

This doesn't cover all cases, but it does cover most of the interesting ones. The first three are the typical ones that appear in physics.

Boundary and initial conditions:

Usually we think of satisfying a PDE only in a particular region in xyz space, for instance in a ball of some radius R. If we denote the region by Ω, typically it is assumed to be an open, connected set with some piecewise smooth boundary $\partial\Omega$. A *boundary condition* is then an additional equation that specifies the value of u and some of its derivatives on the set $\partial\Omega$. For instance,

$$u = f(x, y, z) \text{ on } \partial\Omega$$

or

$$u_x = g(x, y, z) \text{ on } \partial\Omega$$

are boundary conditions.

An *initial* condition, on the other hand, specifies the value of u and some of its derivatives at some initial time t_0 (often $t_0 = 0$). So the following are examples of initial conditions:

$$u(x, y, z, t_0) = f(x, y, z) \text{ on } \Omega$$

or

$$u_t(x, y, z, t_0) = f(x, y, z) \text{ on } \Omega.$$

As an example, consider the 1D wave equation restricted to the interval $[0, L]$. The region of interest is the open interval $\Omega = (0, L)$ with boundary points $x = 0, L$. A typical physical problem is to solve (for $u = u(x, t)$) the equation

$$\begin{aligned}
u_{tt} &= c^2 u_{xx}, && \text{on the region } 0 < x < L, 0 < t \\
u(0, t) &= 0, && \text{a boundary condition} \\
u(L, t) &= 0, && \text{a boundary condition} \\
u(x, 0) &= f(x), && \text{an initial condition, at } t_0 = 0 \\
u_t(x, 0) &= g(x), && \text{an initial condition.}
\end{aligned}$$

Cauchy, Dirichlet, and Neumann conditions:

We will often hear reference to these three types of boundary/initial conditions. So let's make it clear what it is.

The Cauchy condition specifies the values of u and several of its normal derivatives, along some given smooth surface in the coordinate space of all the independent variables (including time). To have any hope of getting a well-posed problem, it is important to get the dimensions right. So, if u is a function of n variables, the surface S should have dimension n − 1 (it is a hypersurface), and if the PDE is order k, the Cauchy data must specify the values of u and its first k − 1 derivatives along the normal to S:

$$u = f_0, u_\eta = f_1, u_{\eta\eta} = f_2, \ldots u_{\eta\ldots\eta} = f_{k-1} \text{ on } S,$$

where $f_0, \ldots, f_{k-1}$ are given functions. Here u_η means the derivative along the normal to the surface. If u is an analytic function, you can consider doing a power series expansion at points along S, using the Cauchy data and PDE to solve for the coefficients in series expansion.[8]

The initial value problem

$$\begin{aligned} u(x,y,z,0) &= f(x,y,z) \text{ for all } x,y,z \\ u_t(x,y,z,0) &= g(x,y,z) \text{ for all } x,y,z \end{aligned}$$

is an example of a Cauchy problem for any second order ODE, with hypersurface $S = \{(x,y,z,t) : t = 0\}$.

It is important that the hypersurface not be a characteristic surface for the Cauchy problem to be solvable. We won't define characteristic surface here; they come from the coefficients of the PDE, and you would notice if you were on one!

The Dirichlet condition specifies the value of u on the boundary $\partial\Omega$ of the region of interest. Think Dirichlet = Data on boundary.
The Neumann condition specifies the value of the normal derivative, u_η, of the boundary $\partial\Omega$.
Think Neumann = Normal derivative on boundary.

Note that $\partial\Omega$ is a hypersurface, and so the Dirichlet and Neumann conditions each specify less information than the Cauchy condition for second order and higher PDEs. It is rather remarkable that for certain elliptic problems, merely Dirichlet or Neumann data alone suffices to solve the problem.

The point of including boundary and initial problems is to force our solutions to be unique, and hopefully well-behaved. Let's look at what it means to pose a good mathematical problem.

Well-posed problems:

We say a mathematical problem is well-posed if it has the following three properties:
1. **Existence:** There exists at least one solution to the problem;
2. **Uniqueness:** There is at most one solution;
3. **Stability:**The unique solution depends in a continuous manner on the data of the problem. A small change in the data leads to only a small changes in the solution.

It is easy enough to illustrate these ideas with the example of solving for x a linear system of equations

$$\mathbf{A}x = y,$$

for given matrix **A** and vector y. If the matrix A is singular, for some inputs y, no solution may exist; for others inputs y there may be multiple solutions. And if **A** is close to singular, a small change in y can lead to a large change in solution x.

To see this in a PDE context, consider the following problem of solving the 1D heat equation in the positive quadrant x, t > 0. We add some reasonable boundary and initial conditions to try to force a unique solution:

$$\begin{aligned} u_t &= u_{xx}, & x > 0, t > 0 \\ u(x,0) &= 0, & x > 0, \text{ a boundary condition} \\ u(0,t) &= 0, & t > 0, \text{ an initial condition.} \end{aligned}$$

The boundary and initial conditions strongly suggest "the solution" is

$$u(x,t) \equiv 0,$$

which is indeed a solution satisfying the BC and IC. But it is not the only solution; for instance, another solution satisfiying that BC and IC is the function

$$u(x,t) = \frac{x}{t^{3/2}} e^{-x^2/4t}.$$

It is easy to check that this function satisfies the PDE in the open quadrant x, t >0 and extends to be zero on both the positive x axis x > 0, and the positive t-axis t > 0. It is curious that by ignoring the behaviour of the function at the origin (x, t) = (0, 0) somehow allows for more than one solution.

One might suppose this is only a mathematical oddity; perhaps one would reject the second solution based on physical grounds. However, keep in mind that many PDE problemsmay be solved numerically: it is unlikely that your numerical method will be smart enough to reject non-physical solutions, without you considering these possibilities.

The heat equation can also be used to illustrate instability in solutions by observing that diffusion processes, when run backwards, tend to be chaotic. But instability can also come up in elliptic equations as well (which we often think of as "nice"). For instance, fix $\varepsilon > 0$ a small parameter and consider Laplace's equation on the upper half plane, with

$$\begin{aligned} u_{xx} + u_{tt} &= 0, & -\infty < x < \infty, t > 0 \\ u(x,0) &= 0 \text{ all } x, \text{ a boundary condition} \\ u_t(x,0) &= \epsilon \sin\frac{x}{\epsilon} \text{ all } x, \text{ a boundary condition.} \end{aligned}$$

This has solution $u(x,t) = \epsilon^2 \sin(x/\epsilon)\sinh(t/\epsilon)$, which gets very large as $\epsilon \to 0$. Compare this with the zero solution $u_0(x,t) \equiv 0$, which is the solution to the problem for $\epsilon = 0$. Thus we have an instability: the input $u_t(x,0) = \epsilon \sin(x/\epsilon)$ goes to zero as $\epsilon \to 0$ but the output does not converge to the zero solution.

Existence and uniqueness theorems:

The first result, the Cauchy-Kowalevski Theorem, tells us that the Cauchy problem is always locally solvable, if all the functions that appear are analytic. The result is usually stated in terms of an initial value problem; the general result follows by transforming the general Cauchy problem, locally, to an initial value problem.

Theorem: (Cauchy-Kowalevski)

If the functions F, f_0, f_1, . . . f_{k-1} are analytic near the origin, then there is a neighbourhood of the origin where the following Cauchy problem (initial value problem) has a unique analytic solution u = u(x, y, z, t):

$$\begin{aligned}\frac{\partial^k u}{\partial t^k}(x,y,z,t) &= F(x,y,z,t,u,u_x,u_y,u_z,...) \qquad \textit{a k-th order PDE} \\ \frac{\partial^j u}{\partial t^j}(x,y,z,0) &= f_j(x,y,z) \qquad \textit{for all } 0 \le j < k.\end{aligned}$$

The statement means to indicate that the function F depends on the independent variable x, y, z, t as well as u and all its partial derivatives up to order k, except for the "distinguished"one $\frac{\partial u^k}{\partial t^k}$. The proof amounts to chasing down some formulas with power series. We've stated the case for (3+1) dimensions, but it is true in other dimensions as well.